TOPOLOGY OPTIMIZATION WITH H-ADAPTIVITY OF THICK PLATES

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ABSTRACT
In this work we propose a new method for the determination of the optimum topology of moderated thick plates. In order to improve the definition of the material/void interface, reduce the effective number of design variables and bound the solution error we employ an h-adaptivity mesh refinement scheme.

We consider the higher order plate theory proposed by Kant [10]. The higher order theory employed in this work gives a cubic axial stress distribution and a quadratic transversal shears stress distribution along the thickness of the plate and considers a three dimensional constitutive equation.

In order to obtain the optimum topology of the structure we make use of the composite material proposed by Chickermane, H. and Gea, H.C. [6]. The microstructure model consists of spherical micro-inclusions embedded in solid material. The formulation of the optimization problem is defined by the minimization of the compliance of the structure subjected to a volume constraint and side constraints. The design variable is then the average density of the material which is considered to be constant within each finite element.

1. INTRODUCTION

The objective of this work is to develop a computational procedure for the determination of the optimum topology of structures and components subjected to mechanical loads. The work is based on the topology optimization model proposed by Bendsoe and Kikuchi [2]. The basic objective is to determine which regions in the domain of the body should have mass or not. In this work the objective is to minimize the compliance of the structure, i.e., the work done by the external forces. We make use of a triangular finite element and employ an h-version of the adaptive finite element modeling, with the aim of improving not only the solution but also the definition of the contour of the optimum layout of the structure. This idea can be illustrated in Fig. 1, where we consider an initial simply connected domain and obtain, after the layout optimization a multiple connected domain.

Here, we define the following sets:
Γ_u - is the part of the boundary with prescribed displacement, i.e., u = u;
$\Gamma_t$ - is the part of the boundary with prescribed traction, i.e., $t = \mathbf{t}$;  
$\Omega$ - is the domain of the body and  
$\mathbf{b}$ - is the body force.

2. FORMULATION OF THE PROBLEM

The problem will be formulated initially as 3D solid and after we will apply the hypothesis of thick plates theories. In order to perform the topology optimization we will employ the composite material approach by considering the material to be a porous material whose microstructure is given by a matrix with a spherical void. The homogenized constitutive equation of the effective material may be fully expressed in terms of the relative density of the composite material.

The general optimization problem may be formulated as:

$$
\min_{\rho} f(u(\rho), \rho),
$$

such that

$$
g_u(u(\rho)) \leq 0 \quad \text{and} \quad g_{\rho}(\rho) \leq 0.
$$

Here $f(u(\rho), \rho)$ is the objective function, $\rho$ - is the vector of the design variables, $u$ - is the displacement field, the Eqs. (2) are the inequality constraints relating the state variable $u(\rho)$, and the design variables.

The state variable $u(\rho) \in \mathcal{K}_{\mathsf{in}_u}$ must satisfy the equilibrium condition, i.e.,

$$
a(u(\rho), v) = l(v) \quad \forall \quad v \in \mathsf{var}_u
$$

such that

$$
a(u, v) = \int_{\Omega} D'' \varepsilon(u) : \varepsilon(v) d\Omega
$$

and

$$
l(v) = \int_{\Omega} b \cdot \nu d\Omega + \int_{\Gamma_t} t \cdot \nu d\Omega
$$

where $\mathcal{K}_{\mathsf{in}_u}$ is the set of admissible displacements, $\mathsf{var}_u$ is the subspace of the admissible variations of the displacement field, $\varepsilon$ is the infinitesimal strain tensor and $D''$ is the effective constitutive equation associated with the composite material.

The layout optimization problem may be formulated as follows:

$$
\min_{\rho} l(u)
$$
with,

- Volume constraint:
  \[
  \int_{\Omega} \rho d\Omega = \alpha V_c
  \]
  where \( V_c \) is volume of the body and \( \alpha \) is a prescribed volume fraction.

- Side constraints:
  the design variable (relative density) is confined to:
  \[
  \rho \in [0,1]
  \]

3. THICK PLATE THEORY

The Eq. (4) may be specialized for thick plates where we make use of the classical higher order theory proposed by Kant \cite{10}. The theory incorporates a displacement field with the following characteristics:

- Quadratic variation of the transversal shear stress;
- Linear variation of the normal deformation;
- Consideration of the three dimensional Hooke Law for the elastic constitutive equation.

3.1. Kinematics hypothesis

This theory considers a displacement field \( \mathbf{U} \), at a point \( q \), is given by:

\[
\mathbf{U}_q = u_x \mathbf{e}_x + u_y \mathbf{e}_y + u_z \mathbf{e}_z
\]

being \( \mathbf{e}_x \), \( \mathbf{e}_y \) and \( \mathbf{e}_z \) the unitary vectors that form the base of the global Cartesian system and

\[
\begin{align*}
  u_x (x, y, z) &= u(x, y) + z \theta_x (x, y) + z^2 \theta'_x (x, y) \\
  u_y (x, y, z) &= v(x, y) - z \theta_y (x, y) - z^2 \theta'_y (x, y) \\
  u_z (x, y, z) &= w(x, y) + z^2 w' (x, y)
\end{align*}
\]

as a result, the components of the infinitesimal strain tensor are given as:

\[
\begin{align*}
  \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\
  \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\
  \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
  \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \\
  \gamma_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\
  \gamma_{zx} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}
\end{align*}
\]

These components may be written as:

\[
\begin{align*}
  \varepsilon_{xx} &= e_{xx} + z \kappa_{xx} + z^3 \kappa''_{xx}, \\
  \varepsilon_{yy} &= e_{yy} + z \kappa_{yy} + z^3 \kappa''_{yy}, \\
  \varepsilon_{zz} &= e_{zz}, \\
  \gamma_{xy} &= (e_{xy} + e_{yx}) + z (\kappa_{xy} + \kappa_{yx}) + z^3 (\kappa''_{xy} + \kappa''_{yx}), \\
  \gamma_{xz} &= \gamma_{zx} + z^2 \gamma''_{zx} + z^2 \gamma''_{xz} \\
  \gamma_{yz} &= \gamma_{zy} + z^2 \gamma''_{zy} + z^2 \gamma''_{yz}
\end{align*}
\]
Integrating the left hand side of Eq. (3) in \( z \), \( z \in (-\frac{h}{2}, \frac{h}{2}) \), \( h \) - the thickness of the plate, we obtain:

\[
\int \{ N \}^T \cdot \{ e \}^d + \{ M \}^T \cdot \{ \kappa \} + \{ M \}^T \cdot \{ \kappa \}^* + \{ Q \}^T \cdot \{ y \} + \{ Q \}^T \cdot \{ y \}^* + N_m e_m dA .
\] (13)

The generalized loads, associated with the theory, can be expressed as:

- The membrane loading, given in force by unit of length (width).
  \[
  \{ N \} = N_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} dz , \quad i, j = x, y ;
  \] (14)

- The bending for unit of length (width).
  \[
  \{ M \} = M_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} z dz \quad \text{and} \quad \{ M \}^* = M_{ij}^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}^* z^* dz , \quad i, j = x, y ;
  \] (15)

- The shear loading for unit length (width).
  \[
  \{ Q \} = Q_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} z dz \quad \text{and} \quad \{ Q \}^* = Q_{ij}^* = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}^* z^* dz , \quad i = x, y ;
  \] (16)

- The normal loading for unit length (width).
  \[
  N_{nn} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} z dz
  \] (17)

Integrating the first part of the right hand side of Eq. (4) in \( z \), \( z \in (-\frac{h}{2}, \frac{h}{2}) \), considering that \( b = \rho g = \text{cte} \), \( g \) - the vectorial acceleration of the gravity, we obtain:

\[
\int_{A} \rho \cdot v dA ;
\] (18)

for the second part, we derive:

\[
\int_{\Gamma} \mathbf{t} \cdot \mathbf{v} d\Gamma = \int_{S} \{ \mathbf{N} \} \cdot \{ \mathbf{u} \} + \{ \mathbf{M} \} \cdot \{ \mathbf{\theta} \} + \{ \mathbf{Q} \} \cdot \{ \mathbf{y} \} + \{ \mathbf{Q} \}^* \cdot \{ \mathbf{y} \}^* + N_m e_m dS.
\] (19)

In order to avoid a possible locking condition, which has been observed for distorted meshes, we make use of a selective reduced integration rule, at the transversal shear terms.

### 4. MICROSTRUCTURE MODEL

The microstructure model proposed, consists of spherical micro-inclusions embedded in solid material. According to Chickermane, H. and Gea, H.C. [6] the homogenized material properties of a composite material with a relative density \( \rho \) is given by:

\[
E(\rho) = \frac{\rho E}{2 - \nu} \quad \text{and} \quad G(\rho) = \frac{8\rho G}{15 - 7\rho},
\] (20)

where we consider the \( \nu \) is Poisson's ratio of the matrix to be given as \( \nu = 1/3 \). The advantage of this method is that it is derived from the rigorous formulation of the theory of composite materials. The derived homogenized constitutive equation is denoted by \( \mathbf{D}^H(\rho) \).
5. MESH REFINEMENT

In order to improve the definition of the contour of the material boundary, reduce the effective number of design variables and bound the solution error we employ a h-adaptivity scheme. The finite element refinement procedure is implemented by using a triangular finite element which has a linear interpolation for the geometry and a quadratic interpolation for the displacement field. The refinement strategy used is illustrated in Fig. 2.a and 2.b,

![Fig. 2.a: Refined element.](image)

![Fig. 2.b: Transition element.](image)

In order to improve the mesh, after the refinement step, we employ a constrained Laplacian smoothing, which is illustrated in Fig. 3. Here, N is the number of adjacent nodes to node $x_n$.

The Laplacian process is conditional since it will only be applied if the mesh quality of the set of elements, as illustrated in Fig. 3, improves. The mesh quality of the set of elements is given by the quality of the worst element in the set. The measure of quality of a given element is given by:

$$x_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

![Fig. 3: Laplacian smoothing.](image)

$$Q = \frac{6A}{\sqrt{3}(l_{max}P)}$$

(21)

where

- $A$ - is the area of the triangle
- $P$ = one half of the perimeter of the triangle
- $l_{max} = \max \{ab, ac, bc\}$ - is the length of the element's largest side.

The refinement strategy employed [7] is described as follows: A given element is refined if:

(i) the measure of the quality of the element is below a given lower bound;
(ii) the element has a side which forms the material boundary of the given topology of the domain;
(iii) the error estimate is larger than a relative prescribed value.

Once these elements are defined we also refine the elements which have two or more neighboring elements to be refined, obtained by the application of the criteria in (i), (ii) and (iii).
6. ERROR ESTIMATOR

Here, we employ the error estimator proposed by Zienkiewics and Zhu [13] and [14]. In this method we derive an improved solution for the strain field, which is denoted by $\varepsilon^*$. This improved solution is obtained by considering a continuous strain field that is interpolated with the same interpolation field used by the displacement field. Thus,

$$\varepsilon^* = \sum_{i=1}^{n} N_i \varepsilon_i^* = \mathbf{N}^T \mathbf{\varepsilon}^* .$$ (22)

In order to determine the nodal values of the improved strain field, $\varepsilon_i^*$, we minimize the least square error with respect to the strains determined by the finite element solution, denoted by $\hat{\varepsilon}$. As a result, we obtain the following relation:

$$\int_{\Omega} N_i (\varepsilon^* - \hat{\varepsilon}) d\Omega = 0, \quad i = 1, \ldots, n .$$ (23)

Once the improved strain field is derived, we may define an estimator of the strain error that is given as:

$$e_c \approx \varepsilon^* - \hat{\varepsilon} .$$ (24)

With this measure we can estimate the energy norm that is given by:

$$\|e\|^2_{\Omega} \approx \int_{\Omega} \left( \varepsilon^* - \hat{\varepsilon} \right)^T \mathbf{D}^H \left( \varepsilon^* - \hat{\varepsilon} \right) d\Omega ,$$ (25)

where $\mathbf{D}^H$ is the homogenized constitutive equation. The approximate solution $\varepsilon^*$ is obtained by a projection technique and is based on the fact that the finite solution $\hat{u}$ is a continuous displacement field, i.e., $\hat{u} \in C^0(\Omega)$ but the stress field $\hat{\varepsilon}$ obtained from $\hat{u}$ is only piece-wise continuous. For this reason, the method is also known as the gradient recovery solution.

6.1. Determination of $\varepsilon^*$

In order to determine $\varepsilon^*$ we employ a least square minimization of the potential $\psi$ defined as:

$$\psi = \int_{\Omega} \left( \varepsilon^* - \hat{\varepsilon} \right)^T \left( \varepsilon^* - \hat{\varepsilon} \right) d\Omega$$ (26)

where we consider $\varepsilon^*$ to be interpolated within each element as $\varepsilon^* = \sum_{i=1}^{n} N_i \varepsilon_i^*$. Where $\varepsilon_i^*$ represents the vector of stress components evaluated at the i-th node of the element, and $N_i$ are the classical interpolation functions. At this point, once $\varepsilon^*$ is determined, we may compute global average error $e_{GE}$, and the element average error $e_E$, as:

$$e_{GE} = \frac{1}{\Omega} \int_{\Omega} \left( \varepsilon^* - \hat{\varepsilon} \right)^T \mathbf{D}^H \left( \varepsilon^* - \hat{\varepsilon} \right) d\Omega$$

$$e_E = \frac{1}{\Omega_E} \int_{\Omega_E} \left( \varepsilon^* - \hat{\varepsilon} \right)^T \mathbf{D}^H \left( \varepsilon^* - \hat{\varepsilon} \right) d\Omega$$ (27)

The strategy adopted to verify if a given element must be refined, due to the error measure criteria, is given by: if $e_E > (1 + \eta) e_{GE}$, with $\eta > 1$, then we refine the element.
DETERMINATION OF THE KKT CONDITIONS

The Lagrangian functional associated with the optimization problem is given by:

\[
L(\rho, u, v, \eta, \lambda, \lambda_i) = l(u) - a(u, v) + l(v) + \eta \left\{ \int_\Omega \rho d\Omega - \alpha V_a \right\} + \\
+ \int_\Omega \lambda_i (\rho_{\text{int}} - \rho)d\Omega + \int_\Omega \lambda_i (\rho - \rho_{\text{sup}})d\Omega , \quad \text{with} \quad \lambda_i \geq 0 \quad \text{and} \quad \lambda_i \geq 0
\]  

(28)

Here, \((v, \eta, \lambda_i, \lambda_i)\) are the Lagrange multipliers associated with the given constraints. The following relations give the Karush-Kuhn-Tucker necessary conditions for the stationarity of the Lagrangian functional at \((\rho, u, v, \eta, \lambda_i, \lambda_i)\):

\[
\int_\Omega \rho d\Omega = aV_a ; \\
\lambda_i \geq 0, \quad \lambda_i (\rho_{\text{int}} - \rho) = 0, \quad \text{and} \quad (\rho_{\text{int}} - \rho) \leq 0 \quad \text{in} \quad \Omega ; \\
\lambda_i \geq 0, \quad \lambda_i (\rho - \rho_{\text{sup}}) = 0, \quad \text{and} \quad (\rho - \rho_{\text{sup}}) \leq 0 \quad \text{in} \quad \Omega ; \\
- \left[ \frac{\partial D^\prime \prime (\rho)}{\partial \rho} \right] \varepsilon(u) \cdot \varepsilon(v) + \eta - \lambda_i + \lambda_i = 0 , \quad \text{in} \quad \Omega \quad \text{and} \\
a(u,v) = l(v) \quad \forall \quad v \in \text{Var}_u .
\]  

(29)

8. NUMERICAL RESULTS

In the discretization of the problem we employ the Galerkin finite element method. We consider that within each finite element the design parameters are constant, i.e., that the material properties of each element are constants. Therefore, for each finite element we have a design variable that is denoted as \((\rho_e)\). In order to solve the discretized optimization problem we make use of the augmented Lagrangian method combined with a bound constrained Truncated-Newton method. The Truncated-Newton method is preconditioned by a limited-memory Quasi-Newton method with a further diagonal scaling.

Problem(1) : The problem consist in the determination of the best layout for the component shown in Fig. 4. The concentrated transversal load \(P=5.0e06\)N on mid of right side of retangular domain. The thickness of the plate is \(h=0.16\)m. We consider the material properties to be: Young Modulus \(E=215.0e09\) N/m\(^2\), Poisson’s ratio \(\nu = 0.3\). The initial mesh has 620 tri6 elements is illustrated in Fig. 5. The initial mesh with the optimum layout for a 60% of material reduction and the definition of the density shade levels are illustrated in Fig. 5. After the solver improves a adaptivity refinement scheme and obtain the others optimum layout for each refined mesh (Fig. 6 for the first adaptive refined mesh and Fig. 7 for the third adaptive refined mesh).

![Fig. 4: Problem definition.](image)
Problem(2): The problem is given by the determination of the optimum layout of a clamped square plate subjected to a concentrated transversal load $P=5.0\times10^7N$ at the center of the plate, the problem is illustred in Fig. 8. The thickness of the plate is $h=0.12m$. Here, we employ an uniform initial mesh with 128 tri6 elements. The material properties are: Young Modulus $E=215.0\times10^9\,N/m^2$, Poisson's ratio $\nu=0.3$. The dimensions of the square plate is $l=2.0m$. The initial mesh with the optimum layout for a 65% of material reduction and the definition of the density shade levels are illustrated in Fig. 9. After the solver improves a adaptivity refinement scheme and obtain the others optimum layout for each refined mesh (Fig. 10 for the first adaptive refined mesh, Fig. 11 for the second adaptive refined mesh and Fig. 12 for the third adaptive refined mesh).

![Fig. 5: Optimum Layout of initial mesh with 620 tri6 elements.](image1)

![Fig. 6: Optimum Layout of first adaptive refined mesh with 1640 tri6 elements.](image2)

![Fig. 7: Optimum Layout of second adaptive refined mesh with 5440 tri6 elements.](image3)

![Fig. 8: Problem definition.](image4)
CONCLUSION

The proposed algorithm has shown to be effective to obtain the optimum layout of the material. The usage of a remeshing procedure is important to accelerate the determination of the solution to the problem. If, for example, we employ an uniform refinement, we obtain a multigrid approach to the problem. This allow us to determine the solution of a larger

Fig. 9: Optimum Layout of initial mesh with 128 tri6 elements.

Fig.10: Optimum Layout of first adaptive refined mesh with 432 tri6 elements.

Fig.11: Optimum Layout of second adaptive refined mesh with 1408 tri6 elements.

Fig.12: Optimum Layout of third adaptive refined mesh with 5032 tri6 elements.

9. CONCLUSION

The proposed algorithm has shown to be effective to obtain the optimum layout of the material. The usage of a remeshing procedure is important to accelerate the determination of the solution to the problem. If, for example, we employ an uniform refinement, we obtain a multigrid approach to the problem. This allow us to determine the solution of a larger
problem using as a starting point, the converged solution obtained with a smaller mesh. This has shown to be effective. The usage of a non uniform refinement has shown to decrease the number of design variables and this decrease becomes even more relevant if we refine the problem with a very large number of elements. Moreover, the introduction of an error estimator was shown to improve the quality of the optimum layout since the remeshing procedure resulted many times in large elements together with very small elements generation large approximation errors for the solution to the problem.

One of the disadvantages of this approach is that we need to determine the stiffness of the elements and in a pixel type of discretization the determination of the stiffness can be implemented in a more effective way. On the other hand, it requires a refined mesh in order to describe the material boundary with some precision.

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