TOPOLOGY OPTIMIZATION OF COMPLIANT MECHANISMS
WITH FATIGUE STRESS CONSTRAINTS

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Abstract. This work presents a procedure to find optimal designs of compliant mechanisms with fatigue failure criteria using the topology optimization method. It’s employed a perturbation technique called integrated $\varepsilon$-relaxation, which modifies the solution spaces and relaxes the pointwise condition using an integrated constraint. Local constraints are imposed using a strategy of adjoint nodal patches. The technique reduces drastically the number of constraints associated to the problem, improving, consequently, the computational performance. The fatigue constraints must satisfy the modified Goodmann’s criterion for simple multiaxial loads (Sine’s method). Microstructure used to describe material properties is SIMP (Solid Isotropic Material with Penalty). In order to describe the proposed procedure, two design problems were solved.

Keywords: Topology optimization, compliant mechanisms, stress constraints, fatigue

1. INTRODUCTION

Compliant mechanisms are a relatively new breed of jointless and monolithic mechanisms that gains its mobility by elastic deformation as opposed to jointed rigid body motions of conventional mechanisms, see Howell (2001). This behavior allows them to actuate in several areas where conventional rigid body mechanisms are inefficient or still incapable to act. However, the development of a generic procedure for design of compliant mechanisms is highly complex and involves continuum mechanics theory and structural optimization methods.

Various formulations have been proposed for the problem of topological synthesis of compliant mechanisms. In this work we employed a distributed spring model at the output port of the elastic body and incorporate fatigue stress constraints into the design problem.
The majority of the topological synthesis works about compliant mechanisms use only volume or mass constraints. However, no realistic design can be obtained if fatigue stress constraints are not considered. By definition, this breed of mechanism gains its mobility from deflection of flexible members. In general, the mobility is required several times and design requirements can be millions of cycles or infinite life. Repeated loads can cause fatigue failure.

2. FORMULATION OF COMPLIANT MECHANISMS

In this study, a distributed spring’s model is adopted in order to avoid creation of singular stress fields and offer a suitable model for representing the physical behavior of compliant mechanisms. We use the Principle of Minimum Potential Energy; see Shames & Dym (1985), to determine the weak formulation of the design problem. So, consider the problem illustrated in Fig. 1:

![Distributed problem definitions for $\tilde{u}_0(\tilde{x})$](image)

For this problem, the total potential energy may be expressed as:

$$\pi(\tilde{u}) = \frac{1}{2} \int_{\Omega} D\varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, d\Omega + \sum_{j=1}^{n} \frac{1}{2} \int_{\Gamma_j} k_j \langle \tilde{u}, \tilde{e}_j \rangle \, d\Gamma - \int_{\Gamma} \langle \tilde{u}, \tilde{e}_m \rangle \, d\Gamma. \tag{1}$$

The three terms of $\pi(\tilde{u})$ are, respectively: the strain energy of body, strain energy of spring and the work done by the external forces. The spring term allows the consideration of several output ports $j = 1, \ldots, n$. Each output means a distributed spring added to the system. In this work, by simplicity, only one output port will be considered. Thus,

$$\pi(\tilde{u}) = \frac{1}{2} \int_{\Omega} D\varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, d\Omega + \frac{1}{2} \int_{\Gamma} k \langle \tilde{u}, \tilde{e}_1 \rangle \, d\Gamma - \int_{\Gamma} \langle \tilde{u}, \tilde{e}_m \rangle \, d\Gamma. \tag{2}$$

Accordingly to the Principle of Minimum Potential Energy, the displacement which minimizes the potential energy is the one that solves the equilibrium problem. Thus, the variational formulation for this problem may be stated as follows:
Find $\ddot{u}_o(\bar{x}) \in \Theta$ so that:

$$\ddot{u}_o(\bar{x}) = \arg \min_{u \in \Theta} \pi(\dot{u})$$

(3)

The necessary optimality condition, for $\ddot{u}_o(\bar{x})$ be a local minimum, is:

$$\frac{d}{d\alpha} \left\{ \pi(\ddot{u}_o + \alpha \ddot{v}) \right\} \bigg|_{\alpha=0} = 0, \ \forall \ddot{v} \in \Theta.$$  \hspace{1cm} (4)

what yields

$$\int_{\Omega} \sigma(\ddot{u}_o) \cdot \varepsilon(\ddot{v}) \, d\Omega + \int_{\Gamma} k_s [\ddot{e}_s \otimes \ddot{e}_s] \ddot{u}_o \ddot{w} \, d\Gamma = \int_{\Gamma} \langle \ddot{v}, \ddot{e}_m \rangle \, d\Gamma, \ \forall \ddot{v} \in \Theta.$$  \hspace{1cm} (5)

It is important observe that $\Delta_s = \langle \ddot{u}_o(\bar{x}), \ddot{e}_s(\bar{x}) \rangle$ represents the displacement at the tip of the spring. Thus, in the case of $\Delta_s \geq 0$, the spring is compressed, and if $\Delta_s < 0$ the spring is elongated. These conventions are illustrated in Figs. 2 (a) and (b).

![Diagram](image)

(a) $\Delta_s = \langle \ddot{u}(x_s), \ddot{e}_s \rangle > 0$  \hspace{1cm} (b) $\Delta_s = \langle \ddot{u}(x_s), \ddot{e}_s \rangle < 0$

Figure 2 – (a) Spring deformation (compression) and (b) spring deformation (traction)

The goal for design of optimal compliant mechanism is maximize the displacement $\Delta_s$, i.e., maximize the component of displacement in the direction of unit vector $\ddot{e}_s$. We can notice that if the movement of the mechanism occurs in the opposite direction of $\ddot{e}_s$, then the elongation of the spring will be $\Delta_s < 0$. On the other hand, if the displacement is in the direction of $\ddot{e}_s$, then $\Delta_s \geq 0$.

3. FATIGUE CONSTRAINTS

The safety region, for a life of $N$ cycles, with $S_f^N$ representing the fatigue limit, is illustrated in Fig. 3 (a).
Figure 3 – (a) Modified Goodman’s diagram and (b) feasible region

Here, $\sigma_y$ is the yielding stress and $\sigma_u$ the ultimate stress. The safety region may be defined by imposing the three constraints, as shown in Fig. 3 (b). Alternating equivalent stress, $S_a$, presents only positive values, since it corresponds to the Von Mises equivalent stress measure (Sine's method). Fatigue constraints $g_1$, $g_2$ and $g_3$ can, then, be written as:

$$g_1(S_a, S_m) = \frac{S_a}{S_f^*} + \frac{S_m}{\sigma_u} \leq 1$$

$$g_2(S_a, S_m) = \frac{S_a}{S_f^*} \leq 1$$

$$g_3(S_a, S_m) = \frac{S_a}{\sigma_y} - \frac{S_m}{\sigma_y} \leq 1$$

In this work we employ Sines' method, see Dowling (1999) or Sines (1959). The Sines's method may be applied by the use of equivalent uniaxial stresses. The amplitude of principal stresses $\sigma_{1a}$, $\sigma_{2a}$ and $\sigma_{3a}$, can be used to evaluate an alternating equivalent stress. For plane stress state ($\sigma_3 = 0$) we have:

$$S_a = \frac{1}{\sqrt{2}} \left( \sigma_{1a}^2 - \sigma_{1a} \sigma_{2a} + \sigma_{2a}^2 \right)^{\frac{1}{2}}$$

We assume that the effective mean stress, $S_m$, and the equivalent alternating stress, $S_a$, are given respectively by:

$$S_m = \text{tr} (\sigma_m)$$

$$S_a = \sigma_{aeq}^{vm}$$

where $\sigma_m = \frac{1}{2} \left( \sigma_{\text{max}} + \sigma_{\text{min}} \right)$ is the mean stress tensor and $\sigma_a = \frac{1}{2} \left( \sigma_{\text{max}} - \sigma_{\text{min}} \right)$ is the alternating stress tensor.
4. DEFINITION OF MATERIAL MODEL

In order to solve the topological optimization problem we used the material approach, which consider a porous material characterized by relative density, \( \rho \). The effective properties of material at intermediate densities are modeled by the SIMP model; see Bendsoe & Sigmund (1999), Duysinx & Sigmund (1998) and Duysinx & Bendsoe (1998). In the SIMP model, the effective Young’s modulus is given, in terms of relative density, as:

\[
E(\rho) = \rho^n E_0.
\]  

(12)

Here, \( E_0 \) is the Young's modulus of the compact material, and \( \eta \) is a regularization parameter. In this work we consider \( \eta = 3 \).

The introduction of an effective stress that penalizes intermediate density regions is necessary to establish a fatigue-stress criterion for this model. This work considers an effective stress proposed by Duysinx & Sigmund (1998) and Duysinx & Bendsoe (1998), which is defined by:

\[
\sigma^*(\rho) = \frac{\sigma(\rho)}{\rho^n}
\]

(13)

With the effective stress defined by Eq. 13, fatigue-stress constraints for SIMP model may be write as:

\[
g_i \left( S_m^*, S_m^* \right) \leq 1 \quad (i=1,2,3)
\]

(14)

with

\[
S_m^*(\rho) = tr \left[ \sigma_m^*(\rho) \right] = tr \left[ \frac{\sigma_m(\rho)}{\rho^n} \right]
\]

(15)

and

\[
S_a^*(\rho) = \frac{\left[ \sigma_a(\rho) \right]_{eq}^{im}}{\rho^n}
\]

(16)

4.1. \( \varepsilon \) - relaxed stress criterion

The major difficulty in the design of compliant mechanisms with fatigue stress constraints is caused by singularity problems caused by the degeneracy of design space. In order to circumvent this problem, we apply a continuation method, denoted the \( \varepsilon \)-relaxation method, see Cheng & Guo (1997). This approach reformulates the singular problem as a sequence of perturbed non-singular problems. As a result of the \( \varepsilon \)-relaxation method, the fatigue stress criterion are modified and given by:

\[
\rho \left( g_i - 1 \right) - \varepsilon \left( 1 - \rho \right) \leq 0
\]

(17)
where \( g_i (S^*_a, S^*_m) \), with \( i=1,2,3 \), are the fatigue-stress constraints for SIMP model.

This relaxation opens the degeneracy region of design space and allows the creation and removal of holes without violating the fatigue-stress constraints.

### 5. DESCRIPTION OF PROBLEM

The objective of the design problem is to find an optimal layout of compliant mechanism defined in a fixed design domain, in which an application of a load or displacement in \( \Gamma_i \), results in a force or displacement on \( \Gamma_i \).

The objective function to be maximized or minimized, for the design of compliant mechanisms, should generate optimal layout that attain conflicting requirements. The optimal layout must be flexible enough to deflect and make contact with the workpiece, and rigid enough to support external loads and transmit force.

#### 5.1. Formulation of the problem

The layout optimization problem for compliant mechanisms subjected to fatigue stress constraints may be formulated as:

\[
\max_{\rho} \int_{\Gamma_i} \{ \tilde{u}_o (\tilde{\rho}), \tilde{e}_i \} d\Gamma
\]

Subjected to the following constraints:

- Local fatigue stress constraints:

  \[
g_i (S^*_a, S^*_m) \leq 1, \quad (i=1,2,3)
\]

- Side constraints:

  \[
  0 \leq \rho \leq 1
\]

where \( \tilde{u}_o (\tilde{\rho}) \) is the solution of:

\[
\int_{\Omega} \sigma (\tilde{u}_o) . \epsilon (\tilde{v}) d\Omega + \int_{\Gamma_j} k_j [\tilde{e}_j \otimes \tilde{e}_j] \tilde{u}_o \tilde{v} d\Gamma = \int_{\Gamma_j} \{ \tilde{v}, \tilde{e}_m \} d\Gamma, \quad \forall \tilde{v} \in \vartheta
\]

#### 5.2. Patch integrated \( \epsilon \)-relaxed stress constraints

Here, it is important to notice that, the relaxed fatigue stress constraints presented in Eq. 17, must be satisfied for all \( \tilde{x} \in \Omega \), characterizing a parametric constraint. One effective method of handling a parametric constraint is to relax the point wise criterion and consider a set of patch integrated constraints. Here, we propose the following patch integrated relaxed fatigue stress constraints: Let \( \Omega_p \) define a node patch, then:

\[
\left\{ \frac{1}{\Omega_p} \int_{\Omega_p} \{ \rho (g_i -1) - \epsilon (1-\rho) \} d\Omega \right\}^1 = 0
\]
where \( g_i \) are the fatigue-stress constraints for SIMP model and \( \langle f(\ddot{x}) \rangle \) defines the positive part of \( f(\ddot{x}) \), i.e., \( \langle f(\ddot{x}) \rangle = \max\{0, f(\ddot{x})\} \).

The introduction of the patch measures, denoted as the \( r \)-mean, reduces considerably the computational efforts at the expense of using a weak form of the \( \epsilon \)-relaxed effective fatigue-stress constraint. The choice of \( r \) must result from a compromise based on numerical results.

### 5.3. Formulation of the Discrete Problem

The discrete formulation for topological optimization problem of compliant mechanisms design subjected to fatigue-stress constraints integrated in node patch, may be stated as:

Determine the relative density field \( \hat{\rho} \in X \), \( X = \{ \tilde{\rho} \in \mathbb{R}^n | \rho_i^{\text{inf}} \leq \rho \leq \rho_i^{\text{sup}} \} \), that solves:

\[
\min f(\hat{\rho}) = \min \left\{ -\frac{1}{\varphi_{sp}} \int_{\Gamma_1} \langle \ddot{u}_o (\hat{\rho}), \vec{e}_s \rangle d\Gamma \right\}
\]

(23)

where the constant \( \varphi_{so} \) is given by:

\[
\varphi_{so} = \int_{\Gamma_1} \langle \ddot{u}_o (\hat{\rho}_o), \vec{e}_s \rangle d\Gamma
\]

(24)

evaluated at the initial relative nodal density \( \hat{\rho}_o \), and subjected to the following constraints:

\[
h_{p-2}(\hat{\rho}, \ddot{u}_o (\hat{\rho})) = \left\{ \frac{1}{\Omega_p} \int_{\Omega_p} \left\langle \rho \left( \frac{S_u^{*}}{S_f} + \frac{S_r^{*}}{\sigma_y} - 1 \right) - \epsilon (1 - \rho) \right\rangle \right\}^{\frac{1}{2}} = 0,
\]

(25)

\[
h_{p-1}(\hat{\rho}, \ddot{u}_o (\hat{\rho})) = \left\{ \frac{1}{\Omega_p} \int_{\Omega_p} \left\langle \rho \left( \frac{S_u^{*}}{S_f} - 1 \right) - \epsilon (1 - \rho) \right\rangle \right\}^{\frac{1}{2}} = 0,
\]

(26)

\[
h_{p}(\hat{\rho}, \ddot{u}_o (\hat{\rho})) = \left\{ \frac{1}{\Omega_p} \int_{\Omega_p} \left\langle \rho \left( \frac{S_u^{*}}{\sigma_y} - \frac{S_r^{*}}{\sigma_y} - 1 \right) - \epsilon (1 - \rho) \right\rangle \right\}^{\frac{1}{2}} = 0,
\]

(27)

for \( p = 1, \ldots, n_p \), where \( n_p \) representing the total number of node patches.

With the aim of solving the discrete optimization problem we apply the Augmented Lagrangian method, reducing the problem to the solution of a sequence of box constrained minimizations. In order to solve the box-constrained problems, we employed a Memoryless Quasi-Newton Method. Details about the optimization procedure employed in this work are found in Bahia (2005).
6. PROBLEM CASES

In order to describe the proposed procedure we solve two problem cases of compliant mechanism design.

Example 1: Displacement inverter mechanism

Figure 4 displays the initial design domain. It is a simple actuator that inverts a force applied at the input port generating a displacement at the output port in an inverse direction. The mechanism is symmetric with respect to the x-axis. Domain is discretized in a mesh with 1800 elements and 900 nodes. Optimal topology obtained is presented in Fig. 5.a. The distribution of the pointwise local failure criterion is shown in Figs. 5.b, 5.c e 5.d. Notice that, a pointwise violation of 3% of e-relaxed stress criterion can be verified.

Figure 4 –Initial (a) and discretized (b) domains of design

Figure 5 –Topology of inverter mechanism (a) and failure functions $h_1$ (b), $h_2$ (c) e $h_3$ (d)
Here, the solution procedure was stopped after ten days. However, an optimal solution may be obtained by using a faster processor or improving the solution algorithms.

Example 2: Crunching mechanism

The initial domain is shown in Fig. 6. This example is similar to the mechanism proposed by Nishiwaki et al. (1998). The design domain is discretized into 1900 elements and 900 nodes. An optimal topology obtained for this mechanism is shown in Fig. 7.a. The result obtained is similar to Nishiwaki et al. (1998). The distribution of the pointwise local failure criterion is shown in Figs. 7.b, 7.c, and 7.d. Notice that, a pointwise violation of 3% of $\varepsilon$-relaxed fatigue-stress criterion can be verified.

Figure 6 –Initial (a) and discretized (b) domains of design

Figure 7 – Topology of crunching mechanism (a) and failure functions $h_1$ (b), $h_2$ (c) e $h_3$ (d).
16. Conclusions

The incorporation of a fatigue-stress failure criterion is fundamental for the design of ready to manufacture compliant mechanism. This work represents a first step in achieving this goal.

The proposed nodal patch strategy has shown to be efficient for the reduction of the computational cost of imposing fatigue-stress constraints. Moreover, the strategy was able to reduce the total number of constraints but also was able to retain enough constraints in order to avoid excessive violation of the pointwise local constraints.

A violating of the pointwise local failure functions was verified in all problem cases designs. However, these violations occurred only in few elements of the mesh. This indicates that the incorporation of an h-adaptive mesh refinement strategy, as done in Costa Jr. (2003), may reduce drastically these local violations. It must be observed that an effective solution for this class of problems is in the scope of the high performance computing.

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