POSITIONAL FINITE ELEMENT METHOD APPLIED TO NONLINEAR GEOMETRIC PLANE TRUSSES

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Abstract. This paper presents the positional nonlinear geometric formulation for plane truss problems in static analysis. The positional formulation presents an alternative approach for nonlinear problems, since it considers nodal positions as variables of the nonlinear system instead of displacements (widely found in literature). The strain determination is done directly from the proposed positional concept. The initial configuration is assumed as the basis of calculation, i.e., Hooke’s law relates reference stress and engineering strain. Some simple numerical examples are shown proving the accuracy of the proposed formulation.

Keywords: Truss, Geometric nonlinearity, finite elements, positional formulation
1 INTRODUCTION

The main sources of nonlinearities in structures are due to geometry and material. This paper is about geometric nonlinearities caused by large deformations in truss. In this case, the equilibrium equations based on initial configuration are not more valid on deformed configuration and the displacement-strain relationship become nonlinear. Material nonlinearities cause nonlinear stress-strain relationships.

This paper presents an analysis of plane truss with large deformations using the positional nonlinear finite element formulation originally proposed by Coda (2003). The standard formulation for finite element method for solids is the displacement method (Bathe, 2006) in which the fundamental unknowns are the displacements of nodes. In the positional formulation, differently, the unknowns are the positions of nodes. The positional formulation uses the Lagrangian description that describes the kinematics of the deformation in terms of a coordinate system, fixed in space. The positional formulation is classified as Total Lagrange Formulation (Bathe, 2006; Wong & Tin-Loi, 1990).

The positional formulation has been applied to several types of structures. Coda and Greco (2004) applied it to dynamic analysis of frames. Marques (2006) and Maciel (2008) presented analyses for plane and three-dimensional solids, respectively. Shells are analysed in Coda and Paccola (2007) e Coda e Paccola (2008). Recently, Greco et al (2012) compared the positional formulation for space trusses to the corotational formulation (Cook, 2001) and the results presented good agreement.

2 FORMULATION

This section describes the bar element of truss using the positional formulation. The bar element only transmits axial forces and it has constant area \( A \). It is shown in Fig. 1. Coordinates \((X_A, Y_A)\) and \((X_B, Y_B)\) represent the initial configuration of the bar element (also known as reference coordinates). Coordinates \((x_a, y_a)\) and \((x_b, y_b)\) represent the deformed configuration of the bar element. The initial (reference) length and deformed bar element are

\[
\begin{align*}
    l_0 &= \sqrt{(X_B - X_A)^2 + (Y_B - Y_A)^2} \\
    l &= \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}
\end{align*}
\]

we may rewrite the deformed coordinates in vector notation as

\[ \mathbf{x} = [x_1, x_2, x_3, x_4]^T \]

where,

\[
\begin{align*}
    x_1 &= x_a, \\
    x_2 &= y_a, \\
    x_3 &= x_b, \\
    x_4 &= y_b
\end{align*}
\]

In order to obtain the finite element equations for bar elements, the principle of minimum potential energy is used. The total potential energy has the form:

\[ \Pi = U - P \]
where $U$ is the strain energy and $P$ is the potential of applied forces, given by

$$ P = \sum F_i x_i $$

where $F_i$ is the applied force in direction of the coordinate $x_i$. The strain energy is given by

$$ U = \int u dV $$

(1)

where $V$ is the initial volume of the bar element and $u$ is the strain energy density. Considering an isotropic, homogeneous and elastic material, according to the Hooke’s law:

$$ u = \frac{E \epsilon^2}{2} $$

(2)

where $E$ is the Young’s modulus and $\epsilon$ is the engineering strain given by

$$ \epsilon = \frac{l - l_0}{l_0} $$

(3)

Substituting Eq.(2) and Eq.(3) into Eq.(1), we have

$$ U = \int_V \frac{E}{2} \left( \frac{l - l_0}{l_0} \right)^2 dV = \frac{EA(l - l_0)^2}{2l_0} $$.  

By principle of minimum potential energy, the equilibrated configuration correspond to minimum value of total potential energy. In other words, equilibrated configuration is obtained by
requiring all directional derivatives to respect to coordinates $x_i$ to vanish:

$$
\frac{\partial \Pi}{\partial x_i} = \frac{\partial U}{\partial x_i} - F_i = 0, \quad \text{for } i = 1, \ldots, 4
$$

(4)

This is a nonlinear system of equations in which may be solved by the Newton-Raphson method. It starts with a first guess $x^{(0)}$ for an approximated solution of the system. Then it find successively better approximations to the system of equations: $x^{(0)}, x^{(1)}, \ldots, x^{(n)}$. The process continues until a sufficiently accurate solution is reached using a convergence criterion such as $||x^{(k+1)} - x^{(k)}|| < \text{tol}$ where tol is a small number. The Newton-Raphson method is described in details in numerical analysis textbooks (Burden & Faires, 2004).

To become suitable to the Newton-Raphson method, the System (4) will be displayed in a different form. Considering

$$
g_i(x) = \frac{\partial \Pi}{\partial x_i} = \frac{\partial U}{\partial x_i} - F_i
$$

(5)

Replacing Eq. (5) into System (4), we have

$$
\begin{align*}
g_1(x) &= 0 \\
g_2(x) &= 0 \\
g_3(x) &= 0 \\
g_4(x) &= 0
\end{align*}
$$

In vector notation, the System (4) has the form

$$
G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \end{bmatrix} = 0
$$

Then the Newton-Raphson method is given by iterative formula:

$$
x^{(k+1)} = x^{(k)} - \left[ G'(x^{(k)}) \right]^{-1} G(x^{(k)})
$$

(6)

where $G'(x^{(k)})$ is the Jacobian matrix of $G$ evaluated at $x^{(k)}$.

In the following steps, we will rewrite the Eq. (6) in terms of gradient and hessian of the energy strain $U$. Note that, by Eq. (5), $G$ is the gradient of total energy potential. In matrix notation:

$$
G = \nabla \Pi = \nabla U - F
$$

(7)
Then the Jacobian $G'$ is given by

$$G' = \nabla \Pi' = (\nabla U - F)' = (\nabla U)'$$

(8)

once $F' = 0$ because $F$ does not depends on $x_i$. Let $H$ be the Hessian of $U$. Hessians and Jacobians are related by equation

$$\nabla U' = H$$

(9)

Replacing Eq.(7), Eq.(8) and Eq. (9) into Eq.(6) we have the formula for Newton-Raphson method in terms of gradient and hessian of $U$:

$$x^{(k+1)} = x^{(k)} - \left[H(x^{(k)})\right]^{-1} \left(\nabla U(x^{(k)}) - F\right)$$

The gradient $\nabla U$ and the hessian $H$ are given by

$$\nabla U = \frac{AE}{l} \nabla l,$$

$$H = \frac{AE}{l} \nabla l \nabla l^T - \frac{AE}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

where $\nabla l$ is the gradient of deformed bar length $l$:

$$\nabla l = \begin{bmatrix} \partial l / \partial x_a \\ \partial l / \partial y_a \\ \partial l / \partial x_b \\ \partial l / \partial y_b \end{bmatrix} = \frac{1}{l} \begin{bmatrix} x_a - x_b \\ y_a - y_b \\ -(x_a - x_b) \\ -(y_a - y_b) \end{bmatrix}$$

A characteristic of the positional formulation is the description of the bar element directly in the global coordinate system and hence it not require the transformation matrix between global and local coordinates found in the displacement formulation of finite element method.

In next section, it will be presented numerical applications using the positional formulation.

3 NUMERICAL APPLICATIONS

3.1 Rod-spring example

The example is taken from (Bonet, Gil & Wood, 2012) and tests the snap-through behavior. The structure is shown in Fig. 2 and contains a rigid rod and a spring of stiffness $k = 4$, 5N/cm².
Consider the force $F$ and the vertical displacement $v$ as positive downward. The analytical solution of this problem, in terms of deformed angle $\theta$, is given by (Bonet, Gil & Wood, 2012) as follows:

\begin{align}
  u &= 10 \cos \theta - 6 \\
  v &= 8 - 10 \sin \theta \\
  F &= k \tan \theta (10 \cos \theta - 6)
\end{align}

Equations (11) and (12) were used for obtaining the analytical solution of force-displacement relationship shown in the Fig. 3. The solid line in Fig. 3 is called equilibrium path. Each point of equilibrium path represents an equilibrium position of the structure or, equivalently, a minimum value of total potential energy. It is worth noting that in some parts of graph there are three different equilibrium positions for same force value.

To obtain the same force-displacement relationship using the positional formulation (described in Section 2) both members were modelled with bar elements. The rigid rod was modelled by a bar element with $E = 1 \times 10^7$ N/cm$^2$ and cross-sectional area $A = 1$ cm$^2$. The spring was modelled by a bar element of equivalent stiffness such that its Young’s modulus is equal to

$$
E = \frac{kl}{A} = \frac{4.5 \times 6}{1} = 27 \text{ N/cm}^2
$$

Two kinds of graphs were yielded with the positional formulation in which closely follows the equilibrium path found by analytical solution. In the first graph, in Fig. 3(a), the force $F$ is incremented from zero until 15 N using a constant increment of 0.75 N. When the force $F$ become greater than the first peak (at point $a$), the equilibrium position jumps far away from peak and snaps
Figure 3: Force-displacement relationship in Rod-spring structure
to the right side of graph (at point b). This behaviour is called snap-through. In the second graph, in Fig. 3(b), the vertical displacement v is incremented from zero until 17 cm using a constant increment of 0.5 cm. Thus, it is possible to obtain, in Fig. 3(b), the intermediate positions that were “jumped” between point a and point b in Fig. 3(a). Point a is a critical point that represents the maximum force that the structure can support before the snap-through. In the intermediate positions, the force F decreases and reverts twice while the displacement continue to increase. However, it is impossible to stay at one of the intermediate positions because are unstable. When the snap-through stops at point b, the structure starts to support additional force.

Next example performs a non-linear analysis of a cantilever truss and the results are compared with a linear analysis.

### 3.2 Cantilever truss example

The 10-bay-cantilever truss, shown in Fig. 4, is taken from (Abrat & Sun, 1983). Its properties are given by

\[
\begin{align*}
A_l &= 80 \times 10^{-6} \text{ m}^2, \quad L_l = 7.5 \text{ m} \\
A_t &= 60 \times 10^{-6} \text{ m}^2, \quad L_t = 5.0 \text{ m} \\
A_d &= 40 \times 10^{-6} \text{ m}^2, \quad E = 7.17 \times 10^{-6} \text{ N/m}^2
\end{align*}
\]

where \( A_l, A_t \) and \( A_d \) are the cross-sectional areas of longitudinal, transverse and diagonal bars, respectively. \( L_l \) and \( L_t \) are the length of longitudinal and transverse bar, respectively. \( E \) is the Young’s modulus.

![Cantilever truss](image)

**Figure 4: Cantilever truss**

Large deformations of the truss under a static concentrated force \( F \) applied at point A were analysed with the positional formulation. Figure 5 shows the vertical and horizontal displacement of the bar \( AB \) in terms of dimensionless displacements given by

\[
\begin{align*}
-\frac{v_A + v_B}{2L} \quad \text{and} \quad 1 + \frac{u_A + u_B}{2L}
\end{align*}
\]

respectively, where \( v_A \) and \( u_A \) are vertical and horizontal displacements for point A and \( v_B \) and \( u_B \) are vertical and horizontal displacements for point B given by

\[
\begin{align*}
v_A &= y_a - Y_A, \quad u_A = x_a - X_A \\
v_B &= y_b - Y_B, \quad u_B = x_b - X_B
\end{align*}
\]
where the coordinates \((X_A, Y_A)\) and \((X_B, Y_B)\) represent the initial configuration of the bar and \((x_a, y_a)\) and \((x_b, y_b)\) the deformed configuration.

![Cantilever Truss](image)

**Figure 5: Displacement at end of truss**

Figure 5 shows large deformations at end of truss in which change significantly the geometry of the structure. These deformations are draw on Fig. 7 that displays large deformed configurations for different concentrated forces.

Figure 6 shows a comparison between nonlinear analysis with positional formulation and linear analysis performed by the standard linear finite element for trusses (Bathe, 2006). The change of geometry is source of nonlinearities such that the deformation can not be approximated satisfactory by linear analysis when the concentrated force \(F\) is large \((F > 1500 \text{ N})\). Figure 6 shows that the structure has a linear behaviour for \(F < 1500 \text{ N}\).

### 4 CONCLUSION

In this paper was presented a formulation for trusses of the positional finite element method. The positional formulation is attractive in terms of computational implementation because it describes the bar element directly in the global coordinate system and hence it not require the global-to-local matrix transformations. The formulation was applied to a simple rod-string structure with snap-through behaviour. The formulation traced the equilibrium path virtually identical to the equilibrium path given by exact solution of this problem. The formulation also performed a nonlinear analysis of a 10-bay-cantilever truss supporting large loads and compared the results with a linear analysis. For this example, the formulation also gives satisfactory results.

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Cantilever Truss (linear vs nonlinear)

**Figure 6:** The comparison between linear and nonlinear analysis for displacement at end of truss

**REFERENCES**


Figure 7: Deformed cantilever trusses

