



## Kaniadakis statistics and the quantum $H$ -theorem

A.P. Santos<sup>a</sup>, R. Silva<sup>a,b,\*</sup>, J.S. Alcaniz<sup>c</sup>, D.H.A.L. Anselmo<sup>a</sup>

<sup>a</sup> Universidade Federal do Rio Grande do Norte, Departamento de Física, Natal-RN, 59072-970, Brazil

<sup>b</sup> Departamento de Física, Universidade do Estado do Rio Grande do Norte, Mossoró-RN, 59610-210, Brazil

<sup>c</sup> Departamento de Astronomia, Observatório Nacional, Rio de Janeiro-RJ, 20921-400, Brazil

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### ABSTRACT

A proof of the quantum  $H$ -theorem in the context of Kaniadakis' entropy concept  $S_{\kappa}^Q$  and a generalization of stosszahlansatz are presented, showing that there exists a quantum version of the second law of thermodynamics consistent with the Kaniadakis statistics. It is also shown that the marginal equilibrium states are described by quantum  $\kappa$ -power law extensions of the Fermi–Dirac and Bose–Einstein distributions.

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Non-Gaussian statistics are generalizations of the standard Boltzmann–Gibbs statistics aiming at describing a number of physical systems that present some restrictions to the applicability of the usual statistical mechanics.

In this concern, the Kaniadakis statistics has been proposed as a real possibility in this direction [1] (see also [2] for other approaches). Recently, efforts on the kinetic foundations of the Kaniadakis statistics have led to a power-law distribution function and the  $\kappa$ -entropy which emerges in the context of the so-called kinetic interaction principle (see Ref. [3] for a review on this subject).

From the mathematical viewpoint, the  $\kappa$ -framework is based on  $\kappa$ -exponential and  $\kappa$ -logarithm functions, defined as

$$\exp_{\kappa}(f) = \left( \sqrt{1 + \kappa^2 f^2} + \kappa f \right)^{1/\kappa}, \quad (1)$$

$$\ln_{\kappa}(f) = \frac{f^{\kappa} - f^{-\kappa}}{2\kappa}, \quad (2)$$

with  $\exp_{\kappa}[\ln_{\kappa}(f)] = f$ . The  $\kappa$ -parameter lies in the interval  $0 \leq \kappa < 1$  and, as the real index  $\kappa \rightarrow 0$ , the above expressions reproduce the usual exponential and logarithmic properties, so that Eqs. (1) and (2) constitute direct generalizations of the usual ex-

ponential and logarithmic functions  $\forall \kappa \neq 0$ . We refer the reader to [4] for some studies on the above approach in both theoretical and experimental contexts. A discussion on the thermodynamical foundations of this formalism can be found in [5] and a study in the perspective of quantum statistics has been presented in [6,7]. Indeed, in order to introduce the effects of the Kaniadakis framework in quantum statistics, it has been considered the power law generalization of Boltzmann factor which is quantified by parameter  $\kappa$  [7]. Physically, the deformation of the Boltzmann factor can be associated with the statistical correlations (for Tsallis statistics, see [14]).

In this Letter, we are particularly interested in thermodynamical aspects, especially in the so-called quantum  $H$ -theorem.<sup>1</sup> The aim of this Letter is to derive a proof of the quantum  $H$ -theorem by including the effects of Kaniadakis statistics on the quantum entropy  $S^Q$ , as well as by introducing statistical correlations on a collisional term of Boltzmann equation, i.e., the  $\kappa$ -generalization of stosszahlansatz. Additionally, we propose that the stationary states of a quantum gas are simply described by a  $\kappa$ -power law extension of the usual Fermi–Dirac and Bose–Einstein distributions.

We start with the main results of the standard  $H$ -theorem in quantum statistical mechanics, namely, the specific functional form

<sup>1</sup> In the quantum domain, the first derivation of  $H$ -theorem was done by Pauli [10], who showed that the change of entropy with time (as a result of molecular collision) provides the equilibrium states which are described by Bose–Einstein and Fermi–Dirac distributions. Some proofs of the  $H$ -theorem, taking into account non-extensive effects under the chaos molecular hypothesis and entropy, have been discussed in the non-relativistic and relativistic domains in Refs. [11–13], as well as in the quantum domain [14].

\* Corresponding author at: Universidade Federal do Rio Grande do Norte, Departamento de Física, Natal-RN, 59072-970, Brazil.

E-mail addresses: alysonpaulo@dfte.ufrn.br (A.P. Santos), raimundosilva@dfte.ufrn.br (R. Silva), alcaniz@on.br (J.S. Alcaniz), doryh@dfte.ufrn.br (D.H.A.L. Anselmo).

for the entropy<sup>2</sup> and the well-known expression for occupation number, which is the rule of counting quantum states in the case of Bose–Einstein and Fermi–Dirac [8]

$$S^Q = - \sum_{\alpha} [n_{\alpha} \ln(n_{\alpha}) \mp (g_{\alpha} \pm n_{\alpha}) \ln(g_{\alpha} \pm n_{\alpha}) \pm g_{\alpha} \ln(g_{\alpha})], \quad (3a)$$

$$n_{\alpha} = \frac{g_{\alpha}}{e^{\gamma + \beta \epsilon_{\alpha}} \mp 1}, \quad (3b)$$

where  $g_{\alpha}$  is the number of states,  $\beta$  inverse of thermal energy and  $\gamma$  a constant, the upper sign refers to bosons, and the lower one to fermions. These two expressions are the main statistical ingredients of the quantum  $H$ -theorem. As is well known, the evolution of  $S^Q$  with time as a result of molecular collisions leads to the occupation number  $n_{\alpha}$ , i.e., this quantity increases with time towards an equilibrium value as a result of molecular collision:

$$\frac{dS^Q}{dt} \geq 0. \quad (4)$$

Let us now consider a spatially homogeneous gas of  $N$  particles (bosons or fermions) enclosed in a volume  $V$ . The state of a quantum gas is characterized by the occupation number  $n_{\alpha}$ . In this case, the time derivative of the occupation number  $n_{\alpha}$  is given by considering collisions of pairs of particles, where a pair of particles goes from a group  $\alpha, \lambda$  to another group  $\mu, \nu$ . The standard theory [8] shows us that the expected number of collisions per unit of time is given by

$$Z_{\alpha\lambda, \mu\nu} = A_{\alpha\lambda, \mu\nu} n_{\alpha} n_{\lambda} (g_{\mu} \pm n_{\mu}) (g_{\nu} \pm n_{\nu}). \quad (5)$$

The collisions in the sample of gas in a condition specified by taking  $n_{\alpha}, n_{\lambda}, n_{\mu}, n_{\nu}$  as the numbers of particles in different possible groups of  $g_{\alpha}, g_{\lambda}, g_{\nu}, g_{\mu}$ , are described by  $Z_{\alpha\lambda, \mu\nu}$ . The coefficient  $A_{\alpha\lambda, \mu\nu}$  must satisfy the relation

$$A_{\alpha\lambda, \mu\nu} = A_{\mu\nu, \alpha\lambda}, \quad (6)$$

which determines the frequency of collisions that are inverse to those considered, in other words, collisions in which particles are thrown from  $\mu, \nu$  to  $\alpha, \lambda$  instead of from  $\alpha, \lambda$  to  $\mu, \nu$ . This coefficient must have a value close to zero for collisions which do not satisfy the energy partition:

$$\epsilon_{\mu} + \epsilon_{\nu} = \epsilon_{\alpha} + \epsilon_{\lambda}. \quad (7)$$

By considering that the temporal evolution of the distribution  $n_{\alpha}$  is affected by  $\kappa$ -statistical correlations introduced in the collisional term through the generalization of stosszahlansatz, we may assume the following quantum  $\kappa$ -transport equation<sup>3</sup>

$$\frac{dn_{\alpha}}{dt} = C_{\kappa}(n_{\alpha}). \quad (8)$$

Here,  $C_{\kappa}$  defines the quantum  $\kappa$ -collisional term. Our main goal is to show that a generalized collisional term  $C_{\kappa}(n_{\alpha})$  leads to a non-negative expression for the time derivative of the  $\kappa$ -entropy (see Eq. (11)), and that it does not vanish unless the distribution function assumes the stationary form associated with quantum  $\kappa$ -distributions [1].

Let us now introduce the  $\kappa$ -collisional term that must lead to a non-negative rate of change of quantum  $\kappa$ -entropy:

$$\begin{aligned} C_{\kappa}(n_{\alpha}) &= - \sum_{\lambda, (\mu\nu)} A_{\alpha\lambda, \mu\nu} \frac{(g_{\mu} \pm n_{\mu})(g_{\nu} \pm n_{\nu})(g_{\alpha} \pm n_{\alpha})(g_{\lambda} \pm n_{\lambda})}{\tilde{n}_{\alpha}} \\ &\quad \times \left( \frac{n_{\alpha}}{g_{\alpha} \pm n_{\alpha}} \overset{\kappa}{\otimes} \frac{n_{\lambda}}{g_{\lambda} \pm n_{\lambda}} \right) \\ &\quad + \sum_{\lambda, (\mu\nu)} A_{\mu\nu, \alpha\lambda} \frac{(g_{\alpha} \pm n_{\alpha})(g_{\lambda} \pm n_{\lambda})(g_{\mu} \pm n_{\mu})(g_{\nu} \pm n_{\nu})}{\tilde{n}_{\alpha}} \\ &\quad \times \left( \frac{n_{\mu}}{g_{\mu} \pm n_{\mu}} \overset{\kappa}{\otimes} \frac{n_{\nu}}{g_{\nu} \pm n_{\nu}} \right). \end{aligned} \quad (9)$$

The sum above spans over all groups  $\lambda$  and also over all pairs of groups  $(\mu\nu)$ . Also, we make a double inclusion of those terms in the summation for which  $\lambda = \alpha$ . In the sum above, the standard product between the distributions is replaced by the  $\kappa$ -generalization from the molecular chaos hypothesis and  $\tilde{n}_{\alpha}$ , defined in Eq. (16), is introduced by mathematical convenience. Note that, in the limit  $\kappa \rightarrow 0$ , the above expression reduces to the standard case [8]

$$\begin{aligned} C_0(n_{\alpha}) &= - \sum_{\lambda, (\mu\nu)} A_{\alpha\lambda, \mu\nu} n_{\alpha} n_{\lambda} (g_{\mu} \pm n_{\mu}) (g_{\nu} \pm n_{\nu}) \\ &\quad + \sum_{\lambda, (\mu\nu)} A_{\mu\nu, \alpha\lambda} n_{\mu} n_{\nu} (g_{\alpha} \pm n_{\alpha}) (g_{\lambda} \pm n_{\lambda}), \end{aligned} \quad (10)$$

with the molecular chaos hypothesis and the standard  $dn_{\alpha}/dt$  readily recovered.

Let us now assume the generalized entropic measure,<sup>4</sup>

$$S_{\kappa}^Q = - \sum_{\alpha} \left\{ n_{\alpha} \ln_{\kappa} \left( \frac{n_{\alpha}}{g_{\alpha} \pm n_{\alpha}} \right) \pm g_{\alpha} \ln_{\kappa} \left( \frac{g_{\alpha}}{g_{\alpha} \pm n_{\alpha}} \right) \right\}, \quad (11)$$

where we use the functionals  $H_{\kappa}^Q = -S_{\kappa}^Q/k_B$  with  $k_B = 1$ . Note once again that, in the limit  $\kappa \rightarrow 0$ , the expression above reduces to the standard case (3).

Before proceeding with our proof of  $H$ -theorem, let us consider some properties of the so-called  $\kappa$ -algebra, i.e., [1]

$$x \overset{\kappa}{\oplus} y := x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2}, \quad (12a)$$

$$x \overset{\kappa}{\oplus} (-y) = x \overset{\kappa}{\ominus} y := x \sqrt{1 + \kappa^2 y^2} - y \sqrt{1 + \kappa^2 x^2}, \quad (12b)$$

so that Eq. (11) is rewritten as

$$\begin{aligned} S_{\kappa}^Q &= - \sum_{\alpha} n_{\alpha} \left[ \ln_{\kappa}(n_{\alpha}) \sqrt{1 + \kappa^2 [\ln_{\kappa}(g_{\alpha} \pm n_{\alpha})]^2} \right. \\ &\quad \left. - \ln_{\kappa}(g_{\alpha} \pm n_{\alpha}) \sqrt{1 + \kappa^2 [\ln_{\kappa} n_{\alpha}]^2} \right] \\ &\quad \mp \sum_{\alpha} g_{\alpha} \left[ \ln_{\kappa}(g_{\alpha}) \sqrt{1 + \kappa^2 [\ln_{\kappa}(g_{\alpha} \pm n_{\alpha})]^2} \right. \\ &\quad \left. - \ln_{\kappa}(g_{\alpha} \pm n_{\alpha}) \sqrt{1 + \kappa^2 [\ln_{\kappa} g_{\alpha}]^2} \right]. \end{aligned} \quad (13)$$

Now, by combining the  $\kappa$ -properties discussed in Ref. [1], i.e.,

$$\ln_{\kappa}(x \overset{\kappa}{\otimes} y) := \ln_{\kappa}(x) + \ln_{\kappa}(y), \quad (14a)$$

$$\ln_{\kappa}(x) \overset{\kappa}{\ominus} \ln_{\kappa}(y) := \ln_{\kappa} \left( \frac{x}{y} \right) \quad (14b)$$

and the expression (13), we obtain the temporal evolution for the  $\kappa$ -entropy:

<sup>2</sup> We assume a gas appropriately specified by regarding the states of energy for a single particle in the container as divided up into groups of  $g_{\alpha}$  neighboring states, and by stating the number of particles  $n_{\alpha}$  assigned to each such group  $\alpha$ .

<sup>3</sup> For similar arguments in the so-called non-extensive Tsallis framework, see Refs. [8,9].

<sup>4</sup> By following, the results of the paper [9], is possible to show the physical viability between the combinatorial structure of  $W_{\pm, \kappa}$  ( $\kappa$ -generalization of Stirling approximation) and the quantum entropy give by Eq. (11), i.e.,  $S_{\kappa}^Q = \ln_{\kappa}(W_{\pm, \kappa}^{\infty})$ .

$$\begin{aligned} \frac{dS_\kappa^Q}{dt} &= \sum_\alpha \tilde{n}_\alpha \left[ \ln_\kappa(n_\alpha) \ominus \ln_\kappa(g_\alpha \pm n_\alpha) \right] \frac{dn_\alpha}{dt} \\ &= \sum_\alpha \left[ \tilde{n}_\alpha \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) \right] \frac{dn_\alpha}{dt}, \end{aligned} \tag{15}$$

where

$$\tilde{n}_\alpha = 1 + \left[ \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) \right]^{-1} (A_\alpha + B_\alpha - D_\alpha) \tag{16}$$

and

$$A_\alpha = \frac{(n_\alpha^\kappa + n_\alpha^{-\kappa})}{2} \sqrt{1 + \kappa^2 [\ln_\kappa(g_\alpha - n_\alpha)]^2}, \tag{17a}$$

$$\begin{aligned} B_\alpha &= \kappa^2 \left[ \frac{(g_\alpha \ln_\kappa(g_\alpha) - n_\alpha \ln_\kappa(n_\alpha)) \ln_{2\kappa}(g_\alpha - n_\alpha)}{(g_\alpha - n_\alpha) \sqrt{1 + \kappa^2 [\ln_\kappa(g_\alpha - n_\alpha)]^2}} \right. \\ &\quad \left. - \frac{\ln_{2\kappa}(n_\alpha) \ln_\kappa(g_\alpha - n_\alpha)}{\sqrt{1 + \kappa^2 (\ln_\kappa(n_\alpha))^2}} \right], \end{aligned} \tag{17b}$$

$$\begin{aligned} D_\alpha &= \left[ \frac{(g_\alpha - n_\alpha)^\kappa + (g_\alpha - n_\alpha)^{-\kappa}}{2(g_\alpha - n_\alpha)} \right] \\ &\quad \times \left[ g_\alpha \sqrt{1 + \kappa^2 (\ln_\kappa(g_\alpha))^2} \right. \\ &\quad \left. - n_\alpha \sqrt{1 + \kappa^2 (\ln_\kappa(n_\alpha))^2} \right]. \end{aligned} \tag{17c}$$

In particular, when  $\kappa \rightarrow 0$ , we have  $\tilde{n}_\alpha = 1$  and the standard calculation is fully recovered.

Now, by substituting (9) into (15), we find

$$\begin{aligned} \frac{dS_\kappa^Q}{dt} &= \sum_\alpha \sum_{\lambda, (\mu\nu)} A_{\alpha\lambda, \mu\nu} (g_\mu \pm n_\mu) (g_\nu \pm n_\nu) (g_\alpha \pm n_\alpha) (g_\lambda \pm n_\lambda) \\ &\quad \times \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \otimes \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) \\ &\quad - \sum_\alpha \sum_{\lambda, (\mu\nu)} A_{\mu\nu, \alpha\lambda} (g_\alpha \pm n_\alpha) (g_\lambda \pm n_\lambda) \\ &\quad \times (g_\mu \pm n_\mu) (g_\nu \pm n_\nu) \\ &\quad \times \left( \frac{n_\mu}{g_\mu \pm n_\mu} \otimes \frac{n_\nu}{g_\nu \pm n_\nu} \right) \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) \end{aligned} \tag{18}$$

with the summations including all groups  $\alpha$  and  $\lambda$  and all pairs of groups  $(\mu\nu)$ .

Following standard lines [8], we rewrite  $dS_\kappa^Q/dt$  in a more symmetrical form. First, note that changing to a summation over all pairs of groups  $(\alpha, \lambda)$  does not affect the value of the sum. This happens because the coefficients satisfy the equality for inverse collisions [see Eq. (6)]. By implementing these operations we have

$$\begin{aligned} \frac{dS_\kappa^Q}{dt} &= \sum_{(\alpha\lambda), (\mu\nu)} A_{\alpha\lambda, \mu\nu} (g_\mu \pm n_\mu) (g_\nu \pm n_\nu) (g_\alpha \pm n_\alpha) (g_\lambda \pm n_\lambda) \\ &\quad \times \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \otimes \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \\ &\quad \times \left[ \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) + \ln_\kappa \left( \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \right] \\ &\quad - \sum_{(\alpha\lambda), (\mu\nu)} A_{\mu\nu, \alpha\lambda} (g_\mu \pm n_\mu) (g_\nu \pm n_\nu) \\ &\quad \times (g_\alpha \pm n_\alpha) (g_\lambda \pm n_\lambda) \left( \frac{n_\mu}{g_\mu \pm n_\mu} \otimes \frac{n_\nu}{g_\nu \pm n_\nu} \right) \\ &\quad \times \left[ \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) + \ln_\kappa \left( \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \right]. \end{aligned} \tag{19}$$

By taking the arithmetical mean of this expression with the equivalent result obtained by interchanging the pair of indices  $(\alpha, \lambda)$  with the pair  $(\mu, \nu)$ , we obtain

$$\begin{aligned} \frac{dS_\kappa^Q}{dt} &= \frac{1}{2} \sum_{(\alpha\lambda), (\mu\nu)} A_{\alpha\lambda, \mu\nu} (g_\mu \pm n_\mu) (g_\nu \pm n_\nu) (g_\alpha \pm n_\alpha) (g_\lambda \pm n_\lambda) \\ &\quad \times \left[ \ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) + \ln_\kappa \left( \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \right. \\ &\quad \left. - \ln_\kappa \left( \frac{n_\mu}{g_\mu \pm n_\mu} \right) - \ln_\kappa \left( \frac{n_\nu}{g_\nu \pm n_\nu} \right) \right] \\ &\quad \times \left[ \frac{n_\alpha}{g_\alpha \pm n_\alpha} \otimes \frac{n_\lambda}{g_\lambda \pm n_\lambda} - \frac{n_\mu}{g_\mu \pm n_\mu} \otimes \frac{n_\nu}{g_\nu \pm n_\nu} \right]. \end{aligned} \tag{20}$$

The summation in the above equation is never negative because the terms  $g_j \pm n_j$  with  $j = \mu, \nu, \alpha, \lambda$  are always positive and  $g_j \geq n_j$  accounts for the Pauli exclusion principle. Note also that by defining

$$X := \frac{n_\alpha}{g_\alpha \pm n_\alpha} \otimes \frac{n_\lambda}{g_\lambda \pm n_\lambda} \tag{21a}$$

and

$$Y := \frac{n_\mu}{g_\mu \pm n_\mu} \otimes \frac{n_\nu}{g_\nu \pm n_\nu}, \tag{21b}$$

we can show that the function

$$\Phi(X, Y) = (X - Y) \cdot (\ln_\kappa(X) - \ln_\kappa(Y)) \tag{22}$$

is also a positive quantity. Therefore, the quantum  $\kappa$ -entropy is an increasing function of time, i.e.,

$$\frac{dS_\kappa^Q}{dt} \geq 0. \tag{23}$$

Therefore, this inequality states that the quantum  $\kappa$ -entropy must be positive or zero, thereby furnishing a quantum derivation of the second law of thermodynamics in the  $\kappa$ -statistic.

In order to make the proof of the quantum  $H$ -theorem consistent with the  $\kappa$ -statistics, let us now calculate the generalized Fermi-Dirac and Bose-Einstein distributions, which recover the stationary distribution previously obtained by a maximization of  $\kappa$ -entropy [4]. As happens in the standard case,  $dS_\kappa^Q/dt = 0$  is a necessary and sufficient condition for local and global equilibrium. From Eq. (20), we note that the condition must be satisfied, if and only if

$$\begin{aligned} &\ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) + \ln_\kappa \left( \frac{n_\lambda}{g_\lambda \pm n_\lambda} \right) \\ &= \ln_\kappa \left( \frac{n_\mu}{g_\mu \pm n_\mu} \right) + \ln_\kappa \left( \frac{n_\nu}{g_\nu \pm n_\nu} \right). \end{aligned} \tag{24}$$

Here, for a null value of this rate of change, the expression Eq. (24) satisfies the energy relation (7) for collisions with appreciable value of  $A_{\alpha\lambda, \mu\nu}$ . As a matter of fact, the above sum of  $\kappa$ -logarithms remains constant during a collision, i.e., it is a summational invariant. In the quantum regime, the solution of these equations is an expression of the form

$$\ln_\kappa \left( \frac{n_\alpha}{g_\alpha \pm n_\alpha} \right) + \gamma + \beta \epsilon_\alpha = 0, \tag{25}$$

where  $\gamma$  and  $\beta$  are constants independent of  $\alpha$ . After some algebra, we may rewrite Eq. (25) as the quantum  $\kappa$ -distribution

$$n_\alpha = \frac{g_\alpha}{\exp_\kappa(\gamma + \beta \epsilon_\alpha) \pm 1}, \tag{26}$$

where  $\exp_\kappa(x)$  is the  $\kappa$ -exponential function defined in Eq. (1). The above expression, which coincides, for  $g_\alpha = 1$ , with the  $\kappa$ -occupa-

tion number derived in Ref. [1,6], seems to be the most general expression which leads to a vanishing rate of change, and clearly reduces to Fermi–Dirac and Bose–Einstein occupation number in the limit  $\kappa \rightarrow 0$ .

Summing up, we have investigated a  $\kappa$ -generalization of the quantum  $H$ -theorem based on the Kaniadakis statistics. We have shown that the Kaniadakis statistics can be extended in order to achieve the occupation numbers concepts of the quantum statistical mechanics. We notice, for  $g_\alpha = 1$ , that the quantum  $\kappa$ -distributions [Eq. (26)] reproduces the result originally obtained in Ref. [1,6]. It is worth mentioning that a different expression for the quantum entropy, given by  $S_\kappa^Q = \int d^n v \sigma_\kappa(f)$ , where  $\sigma_\kappa(f) = - \int df \ln_\kappa(f/(f+1))$  has been studied in Ref. [6] in the context of the maximal entropy principle. In their analysis, the authors calculated the  $\kappa$ -generalization of Bose–Einstein stationary occupation number, which is similar to our expression (26) for  $g_\alpha = 1$ .

Finally, it is worth emphasizing that this work seems to complement a series of studies on the compatibility between Kaniadakis statistics and the Boltzmann  $H$ -theorem and shows, together with Refs. [1,15], that an  $H_\kappa$ -theorem can be derived in non-relativistic, relativistic and also in a quantum regimes. Also, our formalism is very general, since our assumptions can be applied to any ensemble whose quantity  $S_\kappa^Q$  can be defined and calculated within the framework of Kaniadakis statistics, and whose value of equilibrium (steady state) is obtained by letting this system evolve in time.

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